If chance will have me king, why, chance may crown me.

~Shakespeare, *Macbeth*, Act I, Scene 3

Introduction to Counting and Probability

1 Introduction

Counting and probability is often one of the most difficult subjects for math students to master. There are a variety of different techniques that one can use to approach a problem, and determining how to effectively and accurate count the number of ways for something to occur can be quite a challenge. In today's practice, I will explain and walk through problems relating to three different and important techniques; however, the problem sets will not delineate which technique goes with which problem, as that is all part of the difficulty in these problems.

2 Technique 1: Constructive Counting

Constructive counting is as basic as it sounds. The main goal of this technique is to determine how many integers/objects/etc. there are in a set by figuring out how to construct any element in the set. Throughout the way, we keep track of the number of possibilities for each step, then perform the necessary manipulations in order to find the answer to the question. This is essentially what is taught before Precalculus, although the problems can get very difficult.

Example 1 (AIME 2003)**.** *Define a good word as a sequence of letters that consists only of the letters A, B, and C - some of these letters may not appear in the sequence - and in which A is never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. How many seven-letter good words are there?*

Solution. There are three choices for the first letter in the word. For each letter that we choose, there are two possibilities for the next letter. (For example, if the first letter is an A, then the second letter can either be an A or a C.) Similarly, there are two choices for the third letter, and so on. Thus, there are $3 \cdot 2^6 = \boxed{192}$ total seven-letter good words. ■

3 Technique 2: Casework

This technique is the messiest of the three. Basically, it suggests that sometimes, there are multiple distinct cases that behave in their own special ways, and in order to consider all possibilities we have to divide the calculation into several smaller pieces.

Example 2 (AIME 2014)**.** *An urn contains* 4 *green balls and* 6 *blue balls. A second urn contains* 16 *green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is* 0.58*. Find N.*

Solution. We do casework based on which color is picked.

- *CASE 1: Both marbles are green.* Then the requested probability is $\frac{4}{4+6} \cdot \frac{16}{N+16} = \frac{32}{5(N+16)}$.
- *CASE 2: Both marbles are blue.* Then the requested probability is $\frac{6}{4+6} \cdot \frac{N}{N+16} = \frac{3N}{5(N+16)}$.

Adding both cases and setting the resulting expression equal to the desired probability gives the linear equation

$$\frac{32+3N}{5(N+16)} = \frac{58}{100} \implies \frac{3N+32}{N+16} = \frac{29}{10}.$$

Solving for N gives $N = \boxed{144}$. ■

4 Technique 3: Complementary Counting

The third technique is the hardest to spot but also the most versatile. Basically, it suggests that in order to find the number of elements in a set, we can instead find the number of elements not in the set and then subtract from the total. It's easiest to see with an example.

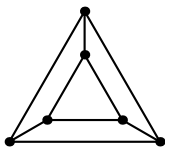
Example 3. *How many ordered (i.e. the order of the books matters) sets of three of the eight Anne of Green Gables books are there if we insist that Anne of Avonela be one?*

Solution. We instead count how many possible ordered sets do not have *Anne of Avonela*. There are seven possibilities for the first book, six for the second, and five for the third, for a total of $7 \times 6 \times 5 = 210$ sets. Since there are $8 \times 7 \times 6 = 336$ possible ordered sets in total, exactly $336 - 210 = \boxed{126}$ of these sets will have them. ■

5 Some More Difficult Problems

In this final section, we conclude with two problems that focus more on combinatorical ingenuity than simple application of the above techniques. (Don't worry, you'll get your chance to test your skills in the Problem Sets.) If these two problems interest you, I encourage you to participate in a round or two of the Mandelbrot competition next year!

Example 4 (Mandelbrot 2013-2014)**.** *In the diagram at right, how many ways are there to color two of the dots red, two of the dots blue, and two of the dots green so that dots of the same color are joined by a segment?*

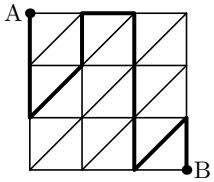


Solution. First, we claim that any valid coloring of the outer equilateral triangle produces a unique coloring for the inner equilateral triangle. There are two distinct cases that can occur:

- *CASE 1: All three points have different colors.* Consider the point that is colored red. There are three segments that connect this point to other points in the diagram, and two of them are colored with different colors. Thus, there exists a unique location for the second red dot: the "corresponding" point on the inner equilateral triangle. Similar reasoning exists for the blue and green dots, and so the claim is proven for this case.
- *CASE 2: Two of the three points have the same color.* Suppose without loss of generality that the common color is red and that the third dot colored is green. By the reasoning in Case 1, there exists a unique position for the green dot. Once this is colored, there are two dots left to color blue, and they are connected by a segment so we are done.

It then suffices to determine the number of ways to color three points with three colors such that no color is applied to more than two points, which is $3^3 - 3 = \boxed{24}$. ■

Example 5 (Mandelbrot 2013-2014)**.** *How many paths are there from A to B through the network shown if you may only move up, down, right, and up-right? A path also may not traverse any portion of the network more than once. A sample path is highlighted.*



Solution. The important part of this problem is that the path can not move to the left in any way, shape, or form. Once it leaves the first column, it can not traverse back there again. Thus, it is advantageous to consider each column independently.

Note that there are seven ways to travel from the first column to the second column: meander through either the three diagonal sides or the four horizontal ones. Once this first column is left, it can be ignored, and the problem is reduced to determining the number of ways to travel from the second column to the third one. But note that this is the exact same problem as before! No matter where the path enters the second column, there are seven ways the path can go to the third column. Similarly, there are seven ways the math can go from the third to the fourth column, at which point the path drops down to B.

This is thus an algorithm to construct any of the $7^3 = \boxed{343}$ possible paths. ■

6 Tips

- **Practice, practice, practice!** As you may have seen, probability problems come in a wide variety of assortments. In order to become good at these types of problems, it helps to develop a sense of intuition, and this can only be achieved through solving problems. Also, when you're done solving a problem, ask yourself which techniques worked and why they worked, as well as which techniques failed and why they did not succeed.
- This is a repeat from last lecture, but **when in doubt, try something!** Great math problems are ones in which the path to the solution is not immediately obvious at first glance. Now, I'm not suggesting to try simply anything but rather work with the end in mind. "What's difficult about this problem? How can I get around this?" (In fact, wishful thinking can often be a huge help this way.)

7 Problems

7.1 Problem Set A

1. [AoPSIntroC&P] How many 4-digit numbers have only odd digits?
2. [AoPSIntroC&P] How many 3-letter words can we make from the letters A, B, C, and D, if we are allowed to repeat letters, and we must use the letter A at least once? **Hint:** 4
3. [AMC 10A 2012] A pair of six-sided fair dice are labeled so that one die has only even numbers (two each of 2, 4, and 6), and the other die has only odd numbers (two each of 1, 3, and 5). The pair of dice is rolled. What is the probability that the sum of the numbers on top of the two dice is 7?
4. [AMC 12B 2006] Mr. and Mrs. Lopez have two children. When they get into their family car, two people sit in the front, and the other two sit in the back. Either Mr. Lopez or Mrs. Lopez must sit in the driver's seat. How many seating arrangements are possible?
5. [Purple Comet HS 2013] How many four-digit positive integers have exactly one digit equal to 1 and exactly one digit equal to 3? **Hint:** 8
6. [AMC 10B 2013] Let S be the set of sides and diagonals of a regular pentagon. A pair of elements of S are selected at random without replacement. What is the probability that the two chosen segments have the same length?
7. [AMC 10A 2013] A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected? **Hint:** 7
8. [AMC 10A 2013] A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?
9. A restaurant has six appetizers, five main courses, and four deserts to choose from its menu. How many possible dinners are there if a main course is required but appetizers and deserts are not? **Hint:** 1

7.2 Problem Set B

10. [AoPSIntroC&P] A senate committee has 5 Republicans and 4 Democrats. In how many ways can the committee members sit in a row of 9 chairs, such that all 4 Democrats sit together? **Hint:** 11
11. [AMC 10A 2013] How many three-digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?
12. In this problem, we calculate the probability that a coin flipped eight times results in exactly five heads.

(a) What is the probability that the sequence **HHHHHTTT** is flipped, where **H** stands for heads and **T** stands for tails?

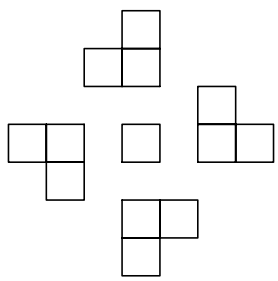
(b) In how many ways is it possible to rearrange the letters in the sequence **HHHHHTTT**? What is the probability of each of these sequences occuring?

(c) What is the probability a coin that is flipped eight times results in exactly five heads?

13. [AIME 2010] Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After waiting at that gate, Dave is told the departure gate has been changed to a different gate, again at random. Let the probability that Dave walks 400 feet or less to the new gate be a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$. **Hint:** 9

14. [AIME 2014] Let the set $S = \{P_1, P_2, \cdots, P_{12}\}$ consist of the twelve vertices of a regular 12-gon. A subset Q of S is called communal if there is a circle such that all points of Q are inside the circle, and all points of S not in Q are outside of the circle. How many communal subsets are there? (Note that the empty set is a communal subset.)

15. [AIME 2013] In the array of 13 squares shown below, 8 squares are colored red, and the remaining 5 squares are colored blue. If one of all possible such colorings is chosen at random, the probability that the chosen colored array appears the same when rotated 90° around the central square is $\frac{1}{n}$, where n is a positive integer. Find n.



7.3 Problem Set C

16. [AMC 12A 2014] A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms? **Hint:** 10

17. [AMC 10A 2009] Three distinct vertices of a cube are chosen at random. What is the probability that the plane determined by these three vertices contains points inside the cube?

★ 18. [AMC 10A 2010] Bernardo randomly picks 3 distinct numbers from the set $\{1,2,3,4,5,6,7,8,9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1,2,3,4,5,6,7,8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number? **Hint:** 3

★ 19. Let m, n, and k be positive integers such that $k \leq \min\{m,n\}$. Prove that

$$\binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \binom{m}{2}\binom{n}{k-2} + \cdots + \binom{m}{k}\binom{n}{0} = \binom{m+n}{k}.$$

This is known as *Vandermonde's Identity.* **Hint:** 5

★ 20. [AMC 12A 2013] Let S be the set $\{1,2,3,...,19\}$. For $a,b \in S$, define $a \succ b$ to mean that either $0 < a-b \leq 9$ or $b-a > 9$. How many ordered triples (x,y,z) of elements of S have the property that $x \succ y$, $y \succ z$, and $z \succ x$? **Hints:** 2, 6

8 Hints

1. The fact that appetizers and deserts may be required means that we can either have an appetizer and a desert or exactly one of the two. What's an easier way to deal with all these cases?
2. Suppose for now that $x > y > z$. Then multiply by an appropriate factor when you're done.
3. The only difference between Bernardo's set of numbers and Silvia's set is that Bernardo's contains a 9. What would the answer be if both sets of numbers were the same? How can you compensate for the extra 9?
4. The phrase "at least" suggests what?
5. It only makes sense that the left hand side represents some sort of casework that alltogether accounts for a "more difficult" way of counting what the right hand side does in one binomial coefficient. Try to figure out a scenario for which this works.
6. The cyclic nature of the conditions suggests that the difference $x - z$ is going to be important. For a fixed difference k, how many places are there for y to "slip through" in between x and z such that both $x \succ y$ and $y \succ z$?
7. First thing's first, how many members are there?
8. This is sort of tricky; remember, a 0 cannot occupy the leftmost digit of any four-digit integer.
9. The edge cases are obviously the most difficult part of this problem. Is there really any nice way to account for those? (Is there?)
10. Apply casework based on how many of the rooms are empty. Be very careful to make sure you're not overcounting; a lot of people got this question wrong on the actual test for that reason!
11. Group the Democrats together into one bundle.