Vieta's Formulas

1 Introduction

In our previous lecture (so long, long, ago...), we explored different types of algebraic manipulations and how having a knowledge of these can help simplify certain problems.¹ In this lesson, we expand this into a class of problems that revolve around Vieta's Formulas, which aren't really formulas in a sense but rather very useful tools for extracting information about the roots of a polynomial without actually knowing the numerical value of the roots themselves.

2 The Quadratic Case

First, we shall explore the case of the general quadratic. This simplest case of Vieta's states the following:

Theorem 1. Let r_1 and r_2 be the roots of the quadratic equation $ax^2 + bx + c = 0$. Then the two identities $r_1 + r_2 = -\frac{b}{a}, \qquad r_1r_2 = \frac{c}{a}$

both hold.

There are two proofs to this, and both are simple. One revolves around the Quadratic formula, while the other involves writing $ax^2 + bx + c = a(x - r_1)(x - r_2) = ax^2 - ax(r_1 + r_2) + ar_1r_2$.

This allows us to find the sum and the product of the roots of any quadratic polynomial without actually computing the roots themselves. (Sounds familiar?)

Example 1. Suppose p and q are the roots of the equation $t^2 - 7t + 5$. Find $p^2 + q^2$.

Solution. Note that from our Vieta's Formulas we have p + q = 7 and pq = 5. Therefore

$$p^{2} + q^{2} = (p+q)^{2} - 2pq = 7^{2} - 2 \cdot 5 = 39$$
.

Example 2. Let m and n be the roots of the equation $2x^2 + 15x + 16 = 0$. What is the value of $\frac{1}{m} + \frac{1}{n}$?

Solution. From Vieta's Formulas, $m + n = -\frac{15}{2}$ and $mn = \frac{16}{2} = 8$. Therefore

$$\frac{1}{m} + \frac{1}{n} = \frac{m+n}{mn} = \frac{-15/2}{8} = \left| -\frac{15}{16} \right|.$$

3 The Cubic Case... and Beyond!

To see how Vieta's Formulas can be expanded beyond quadratics, we look toward the cubic case for help. By using a similar proof as we did in the previous section, we can write

$$x^{3} + bx^{2} + cx + d = (x - r_{1})(x - r_{2})(x - r_{3})$$

= $x^{3} - (r_{1} + r_{2} + r_{3})x^{2} + (r_{1}r_{2} + r_{2}r_{3} + r_{3}r_{1})x - r_{1}r_{2}r_{3}.$

By compensating for the leading coefficient, we get another set of formulas:

Theorem 2. Let r_1, r_2, r_3 be the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$. Then we have

$$r_1 + r_2 + r_3 = -\frac{b}{a},$$
 $r_1r_2 + r_2r_3 + r_3r_1 = \frac{c}{a},$ $r_1r_2r_3 = -\frac{d}{a}.$

 $^{^{1}}$ Considering that the material in that session consists of basically stuff you should know, it should not make a difference on whether or not you attended that lecture.

To put this into words, the first equation deals with the sum of the roots taken one at a time, the second with the sum taken two at a time, and the third taken three at a time (aka the product since there are only three roots). (Side-note: We call each of these expressions on the right-hand sides **symmetric sums**, in that swapping the values of, say, r_1 and r_2 will not affect the value of the whole expression.) Vieta's Formulas for polynomials of degree four or higher are defined similarly, with the first ratio equal to the sum of the roots taken one at a time, the second equal to the sum taken two at a time, the third taken three at a time, and so on.

Problems involving three or more roots are generally more difficult in that more ingenuity/basing capability is needed.

Example 3. Suppose p, q, and r are the roots of the polynomial $t^3 - 2t^2 + 3t - 4$. Find (p+1)(q+1)(r+1).

Solution. As in Example 1, we expand, except this time we have to be more careful:

$$(p+1)(q+1)(r+1) = pqr + pq + qr + rp + p + q + r + 1 = 4 + 3 + 2 + 1 = |10|.$$

Finally, we'll end this lecture with a problem that looks challenging at first sight, but is not that difficult... if you understand Vieta's Formulas.

Example 4 (AoPS). The roots r_1 , r_2 , and r_3 of $x^3 - 2x^2 - 11x + a$ satisfy $r_1 + 2r_2 + 3r_3 = 0$. Find all possible values of a.

Solution. From Vieta's Formulas, we have the system of equations

$$\begin{cases} r_1 + r_2 + r_3 &= 2, \\ r_1 r_2 + r_2 r_3 + r_3 r_1 &= -11, \\ r_1 + 2r_2 + 3r_3 &= 0. \end{cases}$$

Subtracting the first equation from the third one yields $r_2 + 2r_3 = -2$, so $r_2 = 2 - 2r_3$. Plugging this back into the first equation gives $r_1 + (-2 - 2r_3) + r_3 = 2 \implies r_1 = r_3 + 4$. Substituting these expressions for r_1 and r_2 into the second equation gives

$$\begin{aligned} (-2 - 2r_3)(r_3 + 4) + (-2 - 2r_3)r_3 + r_3(r_3 + 4) &= -11\\ (2 + 2r_3)(r_3 + 4) + (2 + 2r_3)r_3 - r_3(r_3 + 4) &= 11\\ (2r_3 + 8 + 2r_3^2 + 8r_3) + (2r_3 + 2r_3^2) - (r_3^2 + 4r_3) &= 11\\ 3r_3^2 + 8r_3 - 3 &= 0\\ (3r_3 - 1)(r_3 + 3) &= 0\\ r_3 &= \frac{1}{3}, -3. \end{aligned}$$

If $r_3 = \frac{1}{3}$, then $r_2 = -2 - 2r_3 = -\frac{8}{3}$ and $r_1 = 4 + r_3 = \frac{13}{3}$, so by Vieta's $a = -r_1r_2r_3 = \boxed{\frac{104}{27}}$. Similarly, if $r_3 = -3$, then we may easily compute $r_2 = 4$ and $r_1 = 1$, so $a = -r_1r_2r_3 = \boxed{12}$. These are our only solutions.

4 Tips

- Learn to recognize certain algebraic manipulations that lead to symmetric sums. Some of these include expanding factored expressions, squaring, or combining fractions under a common denominator.
- Be on the lookout for polynomials where one the coefficients is zero. This does not occur very often, but when it does, it's designed to help simplify the arithmetic and algebra a bit; take it as a blessing!

5 Review Problems

These problems are more geared toward the people who did not attend the previous "Algebraic Manipulations" lecture.

- 1. Let m and n be the two roots of the equation $x^2 15x + 28 = 0$. Find (m+1)(n+1).
- 2. Let $g(x) = x^3 5x^2 + 2x 7$, and let the roots of g(x) be p, q, and r. Compute $p^2qr + pq^2r + pqr^2$.
- 3. [AMC 10A 2003] What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?
- 4. Let r_1 , r_2 , and r_3 be the roots of the polynomial $x^3 14x + 15x 16$. Compute $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_2}$.
- 5. [AMC 10A 2006] Let a and b be the roots of the equation $x^2 mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 px + q = 0$. What is q?
- 6. [Junior Balkan MO 1999] Let a, b, c, x, y be five real numbers such that $a^3 + ax + y = 0$, $b^3 + bx + y = 0$ and $c^3 + cx + y = 0$. If a, b, c are all distinct numbers prove that their sum is zero.
- 7. [Math League HS 2011-2012] If a is real, what is the only real number that could be a multiple root of $x^3 + ax + 1 = 0$?
- 8. [Putnam 1977] Consider all lines that meet the graph of $y = 2x^4 + 7x^3 + 3x 5$ in four distinct points, (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , (x_4, y_4) . Show that

$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

is independent of the line and find its value.

6 Challenge Problems

Good luck.

- 9. The polynomial $x^3 Ax + 15$ has three real roots. Two of these roots sum to 5. What is |A|?
- 10. [Eotvos 1899] Let x_1 and x_2 be the roots of the equation $x^2 (a + d)x + (ad bc) = 0$. Show that x_1^3 and x_2^3 are the roots of the equation

$$y^{2} - (a^{3} + d^{3} + 3abc + 3bcd)y + (ad - bc)^{3} = 0.$$

11. [Purple Comet 2003] Let x_1 and x_2 be the roots of the equation $x^2 + 3x + 1 = 0$. Compute

$$\left(\frac{x_1}{x_2+1}\right)^2 + \left(\frac{x_2}{x_1+1}\right)^2.$$

- 12. [HMMT 2001] Find all the values of m for which the zeros of $2x^2 mx 8$ differ by m 1.
- 13. [Brilliant.org] If the quartic $x^4 + 3x^3 + 11x^2 + 9x + A$ has roots k, l, m, and n such that kl = mn, find A.
- ★ 14. [Purple Comet 2010] Let x_1, x_2 , and x_3 be the roots of the polynomial $x^3 + 3x + 1$. There are relatively prime positive integers m and n such that

$$\frac{m}{n} = \frac{x_1^2}{(5x_2+1)(5x_3+1)} + \frac{x_2^2}{(5x_1+1)(5x_3+1)} + \frac{x_3^2}{(5x_1+1)(5x_2+1)}.$$

Find m + n.

- ★ 15. [USAMO 1984] The product of two of the four roots of the quartic equation $x^4 18x^3 + kx^2 + 200x 1984 = 0$ is -32. Determine the value of k.
- ★ 16. [ISL 1982] Let p(x) be a cubic polynomial with integer coefficients with leading coefficient 1 and with one of its roots equal to the product of the other two. Show that 2p(-1) is a multiple of p(1) + p(-1) 2(1 + p(0)).

7 Answer Key

- 1. 44
- $2.\ 35$
- 3. -1
- 4. $\frac{15}{16}$
- 5. 6
- 6. Note that a, b, and c are the roots of the equation $t^3 + tx + y = 0$, so by Vieta's a + b + c = 0 as desired.
- 7. $\frac{\sqrt[3]{4}}{2}$
- 8. Consider the equation $2x^4 + 7x^3 + 3x 5 = ax + b \implies 2x^4 + 7x^3 + (3 a)x (5 + b) = 0$. By Vieta's, the average of the roots of this equation is constant at $\frac{-7/2}{4} = -\frac{7}{8}$.
- 9. 22
- 10. By Vieta's, $x_1 + x_2 = a + d$ and $x_1x_2 = ad bc$. Then $x_1^3x_2^3 = (ad bc)^3$ and

$${}_{1}^{3} + x_{2}^{3} = (x_{1} + x_{2})^{3} - 3x_{1}x_{2}(x_{1} + x_{2}) = (a+d)^{3} - 3(ad-bc)(a+d) = a^{3} + d^{3} + 3abc + 3bcd.$$

Therefore by again applying Vieta's x_1^3 and x_2^3 are the roots of said quadratic, as desired.

- 11. 18
- 12. $6, -\frac{10}{3}$
- 13. 9
- 14. 3 + 7 = 10
- $15.\ 86$
- 16. Let $p(x) = x^3 + ax^2 + bx + c$. It is not difficult to compute that p(1) = 1 + a + b + c, p(-1) = -1 + a b + c, and p(0) = c. Plugging these expressions into what we want to prove gives that

$$\frac{2p(-1)}{p(1) + p(-1) - 2(1 + p(0))} = \frac{2(-1 + a - b + c)}{(1 + a + b + c) + (-1 + a - b + c) - 2(1 + c)} = \frac{2(a - 1 + c - b)}{2(a - 1)} = 1 + \frac{c - b}{a - 1}.$$

In order for this to be an integer, $\frac{c-b}{a-1}$ must itself be an integer. Let the roots of p(x) be α , β , and $\alpha\beta$. By Vieta's Formulas, we have

$$\begin{cases} \alpha + \beta + \alpha\beta &= -a, \\ \alpha\beta + \alpha^2\beta + \alpha\beta^2 &= b, \\ \alpha^2\beta^2 &= -c. \end{cases}$$

Reckless substitution gives

$$\frac{c-b}{a-1} = \frac{-\alpha^2\beta^2 - \alpha\beta - \alpha^2\beta - \alpha\beta^2}{-\alpha - \beta - \alpha\beta - 1} = \frac{\alpha\beta(\alpha+1)(\beta+1)}{(\alpha+1)(\beta+1)} = \alpha\beta$$

It now suffices to prove that $\alpha\beta$ is an integer. From the third equation we know that $\alpha\beta$ is either an integer or of the form \sqrt{n} , where *n* is an integer that is not a perfect square². Suppose it is of that latter form. From the first and second equations we know that $-a + 1 = \alpha\beta + (\alpha + \beta + 1)$ and $b = \alpha\beta + \alpha^2\beta + \alpha\beta^2 = \alpha\beta(1 + \alpha + \beta)$ are both integers. Denoting $X = \alpha\beta$ and $Y = 1 + \alpha + \beta$, we have that X + Y and XY are both integers. By Vieta's, X and Y are thus the roots of a quadratic equation with integer coefficients, which are known to be of the form $p \pm \sqrt{q}$ for integers p, q. However, since X itself is of the form \sqrt{q} , X and Y must be negatives of each other. But then $\alpha\beta = -(\alpha + \beta + 1) \implies (\alpha + 1)(\beta + 1) = 0$, so either α or β is equal to 1. WLOG let $\alpha = -1$. Then we get $-a = \alpha + \beta + \alpha\beta = -1 + \beta - \beta = -1$, so a = 1. But in this case, our original fraction becomes $\frac{c-b}{0}$, contradiction. Therefore $\alpha\beta$ is indeed an integer.

²Note that n is not necessarily positive, as $\alpha\beta$ can also be imaginary.