Introduction to Conics, Geometrically

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Conics are beautiful. Here's why.

1 Introduction

Conics are a staple of Algebra II and Precalculus curricula in the United States, but their treatments in these courses are somewhat limited. Conics are defined as the graphs of degree-two curves, and the focus[1] of conic sections lies in purely analytic properties and manipulations. However, there is another perspective to conics that is less practical but way more elegant: the synthetic treatment. Instead of defining conics on the coordinate plane, we can define conics in "geometric" ways. While general conics do not have nearly the same potential as circles in general, they still allow for a surprising amount of depth that is worth exploring.

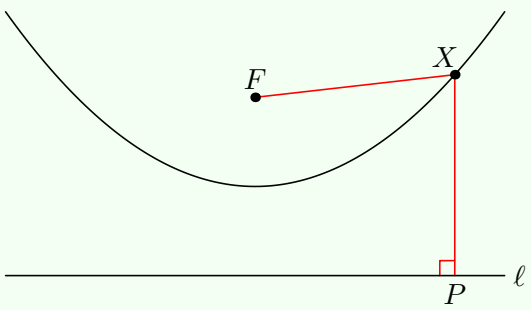
One small thing: this handout will *not* explore conics from a projective viewpoint; there are already a few resources on this topic (mainly in the context of homography or moving points). Instead, we will focus on a purely synthetic treatment of conic sections.

2 Parabolas

2.1 Definitions and Basic Properties

The first conic section we'll cover is arguably the simplest: the parabola.

Definition 2.1. A **parabola** $\mathcal{P}$ is the locus of points X which have equal distances to some fixed point F (called the *focus* of $\mathcal{P}$) and some fixed line ℓ (called the *directrix* of $\mathcal{P}$).



If you haven't seen this definition, it may not be clear that it produces the same curve as the standard $y = ax^2 + bx + c$ definition. Let's quickly verify this before moving on.

[1] pun intended

Proposition 2.1. The parabola with equation $x^2 = 4py$ has focus $(0, p)$ and directrix $y = -p$.

Proof. Remark that $x^2 = 4py$ if and only if

$$\begin{aligned} x^2 + (y-p)^2 &= x^2 + y^2 - 2yp + p^2 \\ &= 4py + y^2 - 2yp + p^2 \\ &= y^2 + 2yp + p^2 = (y+p)^2; \end{aligned}$$

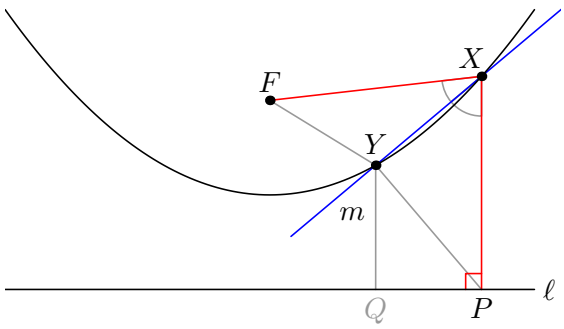
that is, the distance from $(0, p)$ to (x, y) equals $y + p$. The result follows. □

At first glance, it may not be clear how Definition 2.1 allows us to do nontrivial geometry with parabolas. This sentiment is correct, to a degree: parabolas simply don't have the same versatility that circles do. (After all, circles give us uncountably many isosceles triangles for free!)

However, perhaps surprisingly, tangent lines become much more prominent when discussing parabolas (and conics in general). This all starts with the following simple theorem.

Theorem 2.1. Using the notation of Definition 2.1, let m denote the tangent to $\mathcal{P}$ at the point X. Then m bisects $\angle FXP$.

Proof. Suppose the bisector m of $\angle FXP$ is not tangent to $\mathcal{P}$; that is, it intersects $\mathcal{P}$ again at $Y \neq X$. Let Q be the projection of Y onto ℓ. Then $FY = YQ$ by the definition of a parabola while $FY = YP$ since $\triangle FYX \cong \triangle PYX$. But then $PY = YQ$, which is absurd. Therefore m is tangent to $\mathcal{P}$.



□

From this, we get an easy corollary.

Corollary 2.1. The reflection of the focus F over the tangent line m lies on the directrix of $\mathcal{P}$.

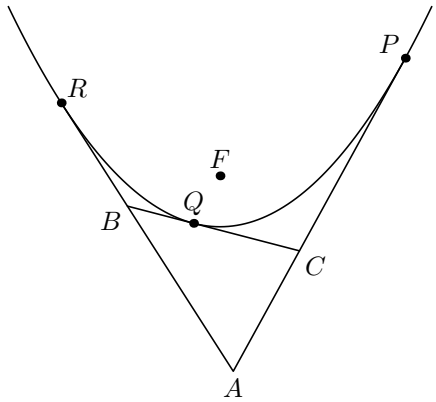
These two results are surprisingly useful, because now we *do* get isosceles triangles for free! They also suggest a general phenomenon when dealing with parabolas: when in doubt, project everything onto the directrix. We illustrate this with a few examples.

Proposition 2.2. Let $\mathcal{P}$ be a parabola with focus F. Fix two points P and Q on $\mathcal{P}$, and suppose the tangents to $\mathcal{P}$ at P and Q intersect at a point R. Further, let X and Y be the projections of P and Q onto the directrix of $\mathcal{P}$. Then $RX = RY$.

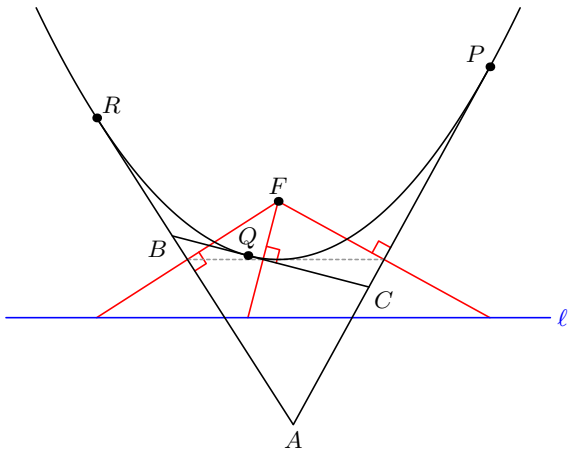
Proof. Dentote by ℓ_P and ℓ_Q the tangent lines to $\mathcal{P}$ at P and Q. Corollary 2.1 implies that ℓ_P is the perpendicular bisector of $\overline{FX}$, and analogously ℓ_Q is the perpendicular bisector of $\overline{FY}$. This means that R, the intersection point of lines ℓ_P and ℓ_Q, is the circumcenter of $\triangle FXY$. The result follows. $\square$

2.2 An Important Configuration

Consider the following scenario, where the pairwise intersection points of three distinct tangents to a parabola $\mathcal{P}$ form a triangle ABC. We wish to find interesting relationships between $\mathcal{P}$ and $\triangle ABC$.

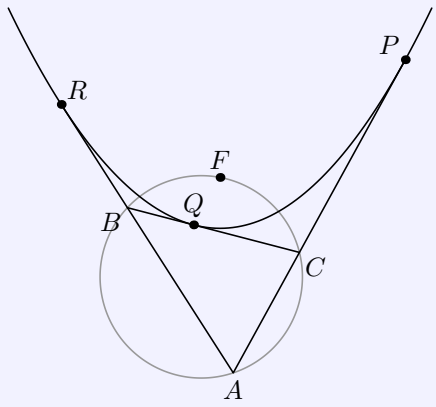


Introduce the directrix ℓ into the picture. The three lines bounding $\triangle ABC$ are each tangents to the parabola $\mathcal{P}$; hence, by applying Corollary 2.1, we know that the reflection of F about each of these tangent lines lies on ℓ. Now considering the homothety $\mathcal{H}$ centered at F with scale factor $\frac{1}{2}$ tells us that the projections of F onto each of these tangent lines are collinear.



But this implies that the dashed line above is the Simson line of F with respect to $\triangle ABC$, giving us the first of two surprising results.

Theorem 2.2. The focus F of $\mathcal{P}$ lies on the circumcircle of $\triangle ABC$.

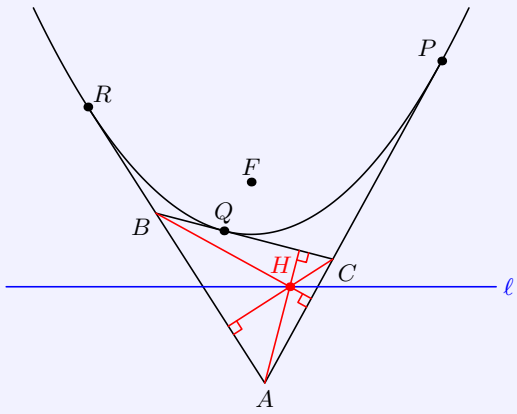


To obtain the second result, recall the following lemma, whose proof can be found in the literature (e.g. page 59 of [2])

Proposition 2.3. Let H be the orthocenter of a triangle ABC, and let P be any point on the circumcircle $\odot(ABC)$. Then the Simson line of P with respect to $\triangle ABC$ bisects the segment $\overline{PH}$.

Now using this proposition and "undoing" the homothety from before leaves us with the following incredible theorem.

Theorem 2.3. The orthocenter H of $\triangle ABC$ lies on the directrix of $\mathcal{P}$.



There are many other theorems that showcase geometric relationships between a parabola and its tangent lines; some of these are deferred to the problems at the end of this handout.

3 Ellipses

The next conic section to discuss is the *ellipse*. Ellipses are slightly more complicated structures to deal with, since they have two foci instead of one. However, this extra structure leads to new properties that make them stand apart from parabolas.

As usual, we start with the definition.

Definition 3.1. An **ellipse** $\mathcal{E}$ is the locus of points P in the plane such that the sum of the distances from P to two fixed points F_1 and F_2 (called the **foci** of $\mathcal{E}$) is a constant.

As with parabolas, there is a simple coordinate representation for ellipses. We record it here for completeness.

Theorem 3.1. Let $\mathcal{E}$ be an ellipse with foci $(c,0)$ and $(-c,0)$. Suppose the sum of the distances from a point (x,y) on $\mathcal{E}$ to the foci is some real number $a > c$. Then $\mathcal{E}$ is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{1}$$

In particular, letting $b^2 := a^2 - c^2$, (1) takes the simple form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Proof. Brute-force computation. The details are neither enlightening nor important for our purposes, so we omit them. □

Unlike parabolas, ellipses have two foci. Several interesting theorems result by considering the interaction between these two points.

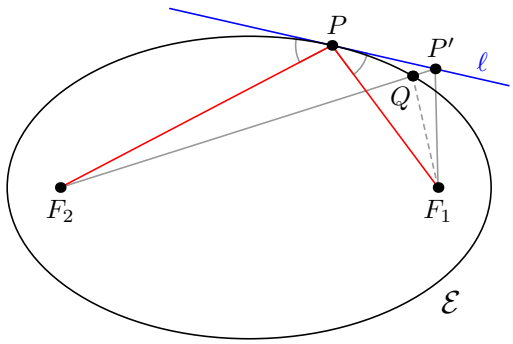
There is a version of Theorem 2.1 that holds for ellipses as well. In order to prove this, we will need the following lemma.

Lemma 3.1. Let ℓ be the tangent to the ellipse at a point P. Then the point P' on ℓ which minimizes the sum $P'F_1 + P'F_2$ is precisely P.

Proof. Let $P' \neq P$ be an arbitrary point on ℓ. Without loss of generality, suppose P' is to the right of P, as shown below. Let $Q = P'F_2 \cap \mathcal{E}$. Then

$$F_1P' + P'F_2 = F_1Q + (QP' + P'F_2) > F_1Q + QF_2 = F_1P + PF_2,$$

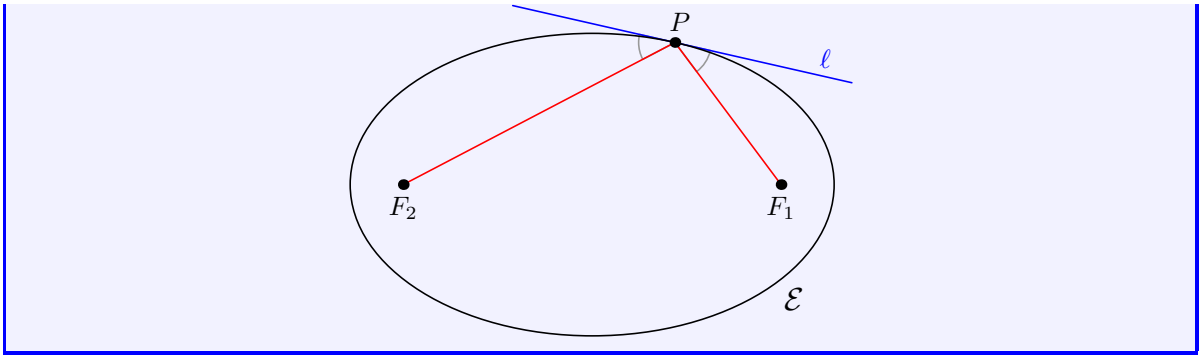
contradicting minimality. □



Remark 1. The proof of Lemma 3.1 does implicitly assume that line ℓ lies completely outside $\mathcal{E}$ – that is, the ellipse $\mathcal{E}$ is *convex*. This is not so hard to prove (with e.g. calculus), but it is not in the spirit of this handout, so we defer its proof to the interested reader.

Lemma 3.1 is the workhorse for the next theorem. Indeed, this lemma gives us one way to characterize the point on ℓ minimizing the sum $PF_1 + PF_2$; to find another way, we need to look to mathematical folklore.

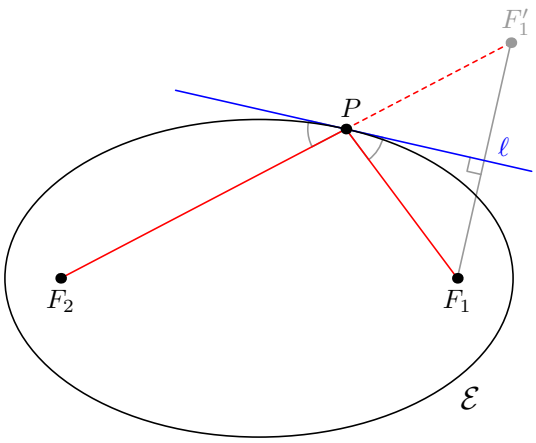
Theorem 3.2. Let ℓ denote the tangent to an ellipse $\mathcal{E}$ at a point P. Then the acute angles made by F_1P and F_2P with respect to ℓ are equal.



Proof. Let F_1' denote the reflection of F_1 about ℓ as shown. Note that for any point $Q \in \ell$,

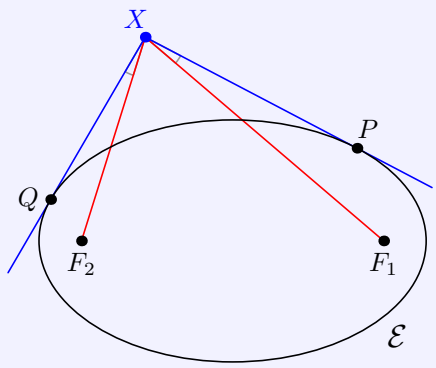
$$F_1Q + F_2Q = F_1Q + QF_1' \geq F_1F_1',$$

with equality when F, Q, and F' are collinear. But recall by the lemma that the point on ℓ which minimizes the sum of the distances to F_1 and F_2 is P. Hence $P = Q$, so F_1', P, and F_2 are collinear. The result follows by vertical angles. □

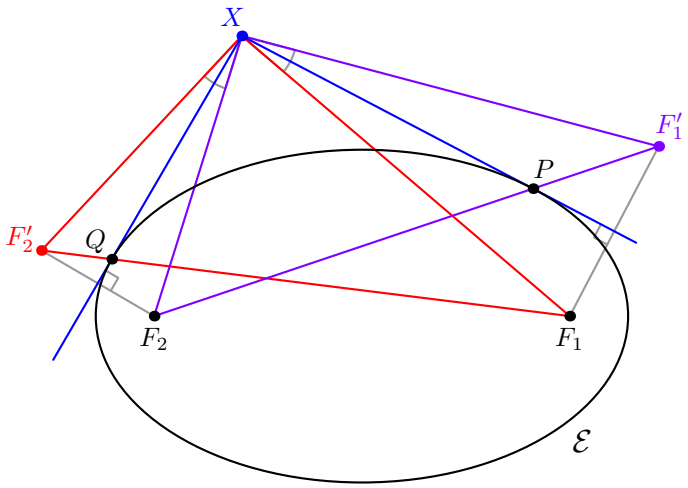


The Reflection Property showcases one way to obtain equal angles from tangents to an ellipse. The next result gives us another way.

Theorem 3.3 (Isogonal Property of Ellipses). Let XP and XQ be tangents to the ellipse $\mathcal{E}$. Then $\angle F_1XP = \angle F_2XQ$.



Proof. Let F_1' denote the reflection of F_1 about XP, and let F_2' denote the reflection of F_2 about XQ.



Observe that $F_1X = F_1'X$ and $F_2X = F_2'X$. Furthermore,

$$F_2F_1' = F_2P + PF_1 = F_2Q + QF_1 = F_1F_2'.$$

Therefore triangles F_1XF_2' and $F_1'XF_2$ are congruent, implying $\angle F_2'XF_1 = \angle F_1'XF_2$. The result follows. □

From this, we get an interesting corollary.

Corollary 3.1. Let F_1 and F_2 be the foci of an ellipse tangent to the sides of a triangle $\mathcal{T}$. Then F_1 and F_2 are isogonal conjugates with respect to $\mathcal{T}$.

Proof. Apply Theorem 3.3 to each of the vertices of $\mathcal{T}$. □

It turns out the opposite is true, too: for any pair of isogonal conjugates $\{P, Q\}$ with respect to a triangle $\mathcal{T}$, there exists an ellipse with foci P and Q tangent to the sides of $\mathcal{T}$. (Think about it!)

Finally, to close this section, we present a problem from a recent Taiwan TST. Even though this last problem does not *really* use ellipses, I would be remiss to include it.

Example 3.1 (Taiwan TST 2019, Evan Chen). Let $\triangle ABC$ be a triangle. Denote its incenter and orthocenter by I and H, respectively. Suppose K is a point in the plane of ABC with

$$AH + AK = BH + BK = CH + CK.$$

Show that H, I, and K are collinear.

Proof. The following solution is due to Telv Cohl.

Let Ω_A be the circle with center A passing through H, and define Ω_B and Ω_C analogously. The given sum condition implies there is a circle Γ with center K that is tangent to Ω_A, Ω_B, and Ω_C internally.

Consider the inversion Φ with center H that fixes $\odot(ABC)$. Remark that circles Ω_B, Ω_C, and Ω intersect at H_A, the reflection of H about $\overline{BC}$. Intersection points H_B and H_C exist similarly. Then Φ sends H_A to the second intersection point of HH_A with $\odot(ABC)$; that is, Φ sends H_A to A. Analogously, Φ sends H_B to B and H_C to C.

Thus Φ sends the circle Ω_A to the line BC, with similar results for the other two circles. But this implies Γ, being tangent to each of Ω_A, Ω_B, and Ω_C, is sent to the incircle of $\triangle ABC$, which has center I. It follows that H, I, and K are collinear.

□

4 Hyperbolas

The final conic section to discuss is the hyperbola. It turns out that hyperbolas and ellipses share many properties, since the synthetic definitions of ellipses and hyperbolas are similar. As a result, we will omit certain proofs if the proof sketches are essentially identical to those in the ellipse case. However, at the end of this section we will briefly touch on some properties of hyperbolas which make them interesting in their own right.

Definition 4.1. A **hyperbola** $\mathcal{H}$ is the locus of points P in the plane such that the positive difference $|PF_1 - PF_2|$ bewteen the distance to two fixed points F_1 and F_2 (called the **foci** of $\mathcal{H}$) is a constant.

The absolute value bars are important for configuration issues, since $\mathcal{H}$ has two branches. On one branch, the difference $PF_1 - PF_2$ is positive; on the other, the difference $PF_2 - PF_1$ is positive.

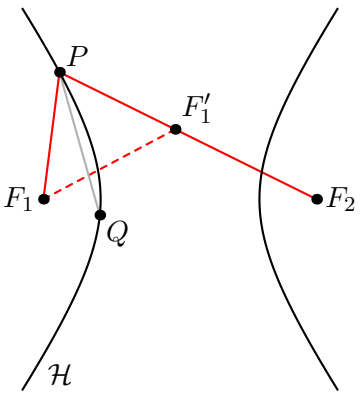
Both ellipses and parabolas have reflection properties. Unsurprisingly, the hyperbola does, too.

Theorem 4.1 (Reflection Property for Hyperbolas)**.** Let $\mathcal{H}$ be a hyperbola with foci F_1 and F_2, and let P be a point on $\mathcal{H}$. Then the tangent to $\mathcal{H}$ at P bisects $\angle F_1PF_2$.

Proof. The proof is kind of a hybrid between the proofs for the other two conic sections. Suppose without loss of generality that $PF_2 > PF_1$. Let ℓ be the angle bisector of $\angle F_1PF_2$, and suppose ℓ intersects the same branch of $\mathcal{H}$ at a point $Q \neq P$. (See the figure below, which is not drawn to scale.) Let F_1' be the reflection of F_1 about ℓ; then F_1' lies on PF_2 and $\triangle PF_1Q \cong \triangle PF_1'Q$. As a result,

$$F_2Q - F_1Q = F_2Q - F_1'Q > F_2F_1' = F_2P - F_1P,$$

where the second step follows from the Triangle Inequality. This is a contradiction, since Q lies on $\mathcal{H}$.



□

It turns out, again unsurprisingly, that the hyperbola also has an isogonal property. Due to configuration issues, it is a bit more clumsy to state and prove, but the idea is still the same.

Theorem 4.2 (Isogonal Property for Hyperbolas)**.** Define $\mathcal{H}$, F_1, and F_2 as usual. Suppose X and Y are two points on $\mathcal{H}$, and let the tangents to $\mathcal{H}$ at X and Y meet at P. Then lines PX and PY are isogonal with respect to $\angle F_1PF_2$.

At this point, one might wonder what hyperbolas can offer that ellipses cannot. The answer lies in a very subtle – yet extremely potent – difference in the coordinate representations between the two conics. Recall Theorem 3.1, which states that the equation for the ellipse has a nice form. The equation for the hyperbola is eerily similar.

Theorem 4.3. For any real numbers a and b, the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

defines a hyperbola centered at the origin with foci on the x-axis. Conversely, any such hyperbola has an equation of this form.

Proof. Analogous to the proof of Theorem 3.1 (which we also omitted, oops...). □

The slight change from a plus sign into a minus sign makes a huge difference in the shape of the graph, because *hyperbolas are not bounded*. In particular, as $|x|$ gets large, we have

$$\frac{y^2}{x^2} = \frac{b^2(\frac{x^2}{a^2} - 1)}{x^2} \to \frac{b^2}{a^2};$$

that is, the points (x, y) approach two lines passing through the origin with slopes $\pm\frac{b}{a}$. These two lines are special enough to warrant names.

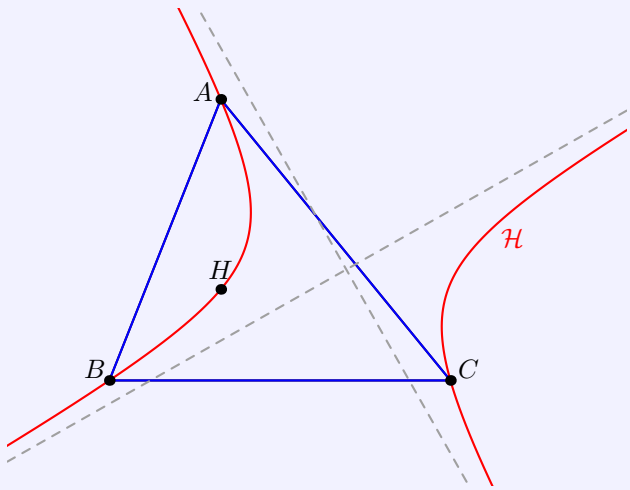
Definition 4.2. These two lines are called the **asymptotes** of $\mathcal{H}$.

In particular, one class of hyperbolas is quite special.

Definition 4.3. A hyperbola is said to be **rectangular** (or **equilateral**) if its asymptotes are perpendicular.

Why are rectangular hyperbolas so special? Most of the theory of rectangular hyperbolas comes down to the following magical theorem. Unfortunately, this theorem is too hard to prove in this handout – the simplest way to prove this uses *projective geometry*. (More on this in the remark to follow.)

Theorem 4.4. Suppose A, B, and C are three points on a hyperbola $\mathcal{H}$, and let H be the orthocenter of $\triangle ABC$. Then H lies on $\mathcal{H}$ *if and only if* $\mathcal{H}$ is rectangular.



Remark 2. Our good friend H returns! This is the second theorem in this handout which notes a surprising property of the orthocenter; the first was Theorem 2.3. This is no coincidence – these two theorems are actually *equivalent*! More specifically, taking the *pole-polar dual* of Theorem 2.3 with respect to a circle centered at H yields the result above. (Try to see why! You'll need the fact that a line tangent to a conic transforms into a point passing through the conic under the pole-polar dual.)

This might shed some light as to why we haven't talked about hyperbolas much here – most of the important facts about hyperbolas are only accessible through **projective geometry**. In projective geometry, we add a so-called **point at infinity** in the direction of each line ℓ in the Euclidean plane. This means, for example, that any two parallel lines actually intersect at a corresponding point at infinity. Another example of this phenomenon is that *a hyperbola passes through the points at infinity in the direction of its asymptotes*. This allows us to treat the asymptotes as actual "points" on the hyperbola, which helps us come to terms with how the asymptotes behave.

5 Problems

Some of these problems require nothing more than the basic definitions, while some require a bit more theory. Which problems are which? You'll find out soon :)

1. Let $\mathcal{P}$ be a parabola with focus F and directrix ℓ. A line through F intersects $\mathcal{P}$ at points A and B and the directrix at point C. If $AF = 3$ and $FB = 2$, compute BC.

2. Let $\mathcal{E}$ be an ellipse with foci F_1 and F_2. Point P lies outside $\mathcal{E}$. The two tangents to $\mathcal{E}$ through P intersect the ellipse at points X and Y, respectively. Prove that F_1P bisects $\angle XF_1Y$.

3. (AMC 12A 2011) A segment through the focus F of a parabola with vertex V is perpendicular to $\overline{FV}$ and intersects the parabola in points A and B. What is $\cos(\angle AVB)$?

4. (Carnegie Mellon 2021, Sam Delatore) Let $f(x) = \frac{x^2}{8}$. Starting at the point $(7, 3)$, what is the length of the shortest path that touches the graph of f and then touches the x-axis?

5. (San Diego 2020-2021) Let ABC be an acute, scalene triangle, and let P be an arbitrary point in its interior. Let $\mathcal{P}_A$ be the parabola with focus P and directrix BC, and define $\mathcal{P}_B$ and $\mathcal{P}_C$ similarly. Suppose $\mathcal{P}_B$ and $\mathcal{P}_C$ intersect at two points; let ℓ_A be the line passing through these two points. Define ℓ_B and ℓ_C similarly (under the same assumption). Show that ℓ_A, ℓ_B, and ℓ_C concur.

6. (Romania TST 2015) Let ABC and ABD be coplanar triangles with equal perimeters. The internal angle bisectors of $\angle CAD$ and $\angle CBD$ meet at P. Show that $\angle APC = \angle BPD$.

7. (Mandelbrot 2013-2014) Suppose an ellipse and a hyperbola have the same points F_1 and F_2 as foci. These curves cross at four points; let P be one of them. These curves also intersect line F_1F_2 at four points Q, R, S and T in this order. Given that $RS = 20$, $ST = 14$, and $\triangle PF_1F_2$ is isosceles, find the area of $\triangle PF_1F_2$.

8. (IMC 2008) Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

9. (Besant) Let $\mathcal{P}$ be a parabola with vertex V and focus F. Fix a point P on $\mathcal{P}$. Suppose the tangent to $\mathcal{P}$ intersects FV at T, while N is the foot of the altitude from P to AV. Prove that V is the midpoint of $\overline{NT}$.

10. (Carnegie Mellon 2020, Own) Let $\mathcal{E}$ be an ellipse with foci F_1 and F_2. Parabola $\mathcal{P}$, having

vertex F_1 and focus F_2, intersects $\mathcal{E}$ at two points X and Y. Suppose the tangents to $\mathcal{E}$ at X and Y intersect on the directrix of $\mathcal{P}$. Compute the eccentricity of $\mathcal{E}$.

11. (AIME 1985) An ellipse has foci at $(9,20)$ and $(49,55)$ in the xy-plane and is tangent to the x-axis. What is the length of its major axis?

12. (AMC 12A 2021, Own) Suppose that on a parabola with vertex V and focus F there exists a point A such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length FV?

13. (Berkeley 2013) A parabola has focus F and vertex V, where $VF = 10$. Let AB be a chord of length 100 that passes through F. Determine the area of $\triangle VAB$.

14. ([3]) Let $\mathcal{E}$ be an ellipse with center O, and P a point outside of $\mathcal{E}$. Prove that the locus of points P for which the tangents from P to the ellipse are perpendicular is a circle centered at O.

15. (Tovi Wen) Let $\mathcal{E}$ be an ellipse with a major axis of length 10. Circle ω with center O is tangent to $\mathcal{E}$ at P. The line OP intersects the major and minor axes of $\mathcal{E}$ at X and Y, respectively. Suppose that $PX = 4$ and $PY = 6$. Find the distance between the foci of $\mathcal{E}$.

16. (Mandelbrot 2005-2006) Triangle ABC is situated within an ellipse whose major and minor axes have lengths 10 and 8. Point A is located at one focus, point B is located at an endpoint of the minor axis, and point C is located on the ellipse so that the other focus lies on $\overline{BC}$. Determine the inradius of $\triangle ABC$.

17. (AIME 2022, Michael Tang) Let a, b, x, and y be real numbers with $a > 4$ and $b > 1$ such that
$$\frac{x^2}{a^2} + \frac{y^2}{a^2-16} = \frac{(x-20)^2}{b^2-1} + \frac{(y-11)^2}{b^2} = 1.$$
Find the least possible value of $a+b$.

18. (KöMaL A.599, November 2013) Two parabolas, P_1 and P_2, have the same focus. The directrix of P_1 meets P_2 at points A and B. The directrix of P_2 meets P_1 at C and D. Show that the points A, B, C, and D are concyclic.

19. (Carnegie Mellon 2017, with Evan Chen) In triangle ABC with $AB = 23$, $AC = 27$, and $BC = 20$, let D be the foot of the A altitude. Let $\mathcal{P}$ be the parabola with focus A passing through B and C, and denote by T the intersection point of AD with the directrix of $\mathcal{P}$. Determine the value of $DT^2 - DA^2$.

20. (Ray Li) The point $(10,26)$ is a focus of a non-degenerate ellipse tangent to the positive x and y axes. What is the line that the locus of the center of ellipse lies on?

21. Let $\mathcal{H}$ be a rectangular hyperbola passing through the vertices of a triangle $\mathcal{T}$. Prove that the center of $\mathcal{H}$ (that is, the intersection of the asymptotes) lies on the nine-point circle of $\mathcal{T}$.

22. (ADMC Round 2, Evan Chang) Two circles with radius 2 centered at points A and B are externally tangent. An ellipse Γ, tangent to both circles, has foci A and B. Points G and H lie on Γ with $AH < AG$ such that G, B, and H are collinear. The tangents to Γ at G and H intersect at P, and AP intersects the angle bisector of $\angle AHB$ at Q. If the line through Q perpendicular to line GH is tangent to the circle centered at B, then find GH.

23. (AoPS) A circle passes through the points $(2,0)$ and $(18,0)$ and is tangent to the parabola whose equation is $y = x^2$. What are the coordinates of the center of the circle?

24. ([1]) Let $\mathcal{P}$ be a parabola with focus F. Let A and B be points on $\mathcal{P}$, and suppose the tangents to $\mathcal{P}$ at A and B meet at a point P. Denote by O the circumcenter of $\triangle PAB$.

Prove that $PF \perp FO$.

25. (Based on Tournament of Towns 2008)[2] Let ω_1 and ω_2 be two disjoint circles, one lying outside of the other. Consider all lines that cut off from the circles two chords of equal lengths. Prove that all such lines are tangent to a fixed parabola.

26. (Ankan Bhattacharya) Let $ABCD$ be a quadrilateral. Suppose there exist a parabola Γ_a with focus A tangent to lines BC, BD, and CD, and a parabola Γ_c with focus C tangent to lines AB, AD, and BD.

    Let X and Y be the tangency points of Γ_a and Γ_c respectively with BD. Prove that $BX = DY$.

27. (Carnegie Mellon 2018, Gunmay Handa) Suppose $\mathcal{E}_1 \neq \mathcal{E}_2$ are two intersecting ellipses with a common focus X; let the common external tangents of $\mathcal{E}_1$ and $\mathcal{E}_2$ intersect at a point Y. Further suppose that X_1 and X_2 are the other foci of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, such that $X_1 \in \mathcal{E}_2$ and $X_2 \in \mathcal{E}_1$. If $X_1X_2 = 8, XX_2 = 7$, and $XX_1 = 9$, what is XY^2?

References

[1] *Conic Sections: Treated Geometrically*, by W.H. Besant, 9th ed., George Bell and Sons, 1895.

[2] *Euclidean Geometry in Mathematical Olympiads*, by Evan Chen, 1st ed., Mathematical Association of America, 2016.

[3] *Geometry of Conics*, by A. V. Akopyan and A. A. Zaslavsky, American Mathematical Society, 2007.

[2] The Tournament of Towns problem gives a bit of a spoiler for this one. Feel free to find that problem if you need a hint.